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Applied Mathematics Letters 13 (2000) 129–135

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Oscillation of Second-Order Linear Difference Equations

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(Received and accepted July 1998)

Communicated by R. P. Agarwal

Abstract—In this paper we give new oscillation and nonoscillation criteria for the self-adjoint difference equation $\Delta(c_{n-1}\Delta x_{n-1}) + a_n x_n = 0$ in terms of the sequence $\sqrt{q}_n = c_n/\sqrt{b_n b_{n+1}}$ where $b_n = c_n + c_{n-1} - a_n$. A necessary and sufficient condition for oscillation is given in the case of q_n periodic with period 2.

Keywords—Linear difference equations, Oscillation and nonoscillation.

1. INTRODUCTION

In this paper, we give some new oscillation and nonoscillation criteria for the self-adjoint difference equation

$$\Delta(c_{n-1}\Delta x_{n-1}) + a_n x_n = 0 \tag{1}$$

with $c_n > 0$.

A solution $\{x_n\}$ of (1) is called nonoscillatory if there is an integer n_0 such that $x_n x_{n+1} > 0$ for all $n \geq n_0$. It is known [1–4] that if one solution of (1) is nonoscillatory, then so is any other nontrivial one. Therefore, (1) can be classified as either oscillatory or nonoscillatory. Oscillation can also be defined in terms of nodes [4, p. 221]. Equation (1) is equivalent to the equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \tag{2}$$

where $b_n = c_n + c_{n-1} - a_n$.

Elementary consideration of signs in (2) implies that it is necessary for (2) to have a nonoscillatory solution is that $b_n > 0$ for large n. So, without any loss of generality, we may suppose that $b_n > 0$ for $n \geq 1$.

In several papers [5–7] and monographs [1–3], the authors gave criteria for oscillation and nonoscillation of (2) in terms of the sequence $q_n = c_n^2/b_n b_{n+1}$. We believe that oscillation and nonoscillation of (2) depends on the ratios a_n/c_n and c_{n-1}/c_n. Unfortunately, equations of

0893-9659/1999/$ - see front matter. Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0893-9659(99)00156-1

type (1), in contrast with the situation in self-adjoint differential equations, cannot be transformed to another one with $c_n \equiv 1$, namely, an equation of the form

$$\Delta^2 x_{n-1} + A_n x_n = 0 \tag{3}$$

with rather 'simple' coefficient A_n. In [5], the authors gave a comparison theorem that connects equations of type (1) with equations of type (3). Here, we give another comparison theorem of that sort, namely Theorem 3.3.

Most of the criteria introduced here depend on the sequence $\{\sqrt{q}_n\}$ and, in particular, the sequence $\{\sqrt{q}_n + \sqrt{q}_{n+1}\}$ and a discussion is made to clarify that our criteria improves the known ones depending on q_n.

Before introducing the main results, we state some of the known results concerning oscillation and nonoscillation of (2). The reader is referred to [1,2,5–7] for proofs and also more other criteria.

THEOREM 1.1. *Equation (2) is nonoscillatory if and only if any of the two first-order equations*

$$c_n r_n + \frac{c_{n-1}}{r_{n-1}} = b_n \tag{4}$$

and

$$q_n s_n + \frac{1}{s_{n-1}} = 1 \tag{5}$$

has an eventually positive solution.

THEOREM 1.2. *Equation (2) is oscillatory if $q_n \geq 1 + \epsilon$ for some $\epsilon > 0$ and nonoscillatory if $4q_n \leq 1$, $(n \geq n_0)$.*

THEOREM 1.3. *If equation (2) is nonoscillatory, then there is an integer N such that for all $m \geq N$ and $k \geq 0$,*

$$4^k q_m q_{m+1} \cdots q_{m+k} < 1. \tag{6}$$

2. OSCILLATION AND NONOSCILLATION CRITERIA

The following theorem generalizes Theorem 1.1.

THEOREM 2.1. *Equation (2) is nonoscillatory if and only if there is an eventually positive sequence $\{\xi_n\}$ such that*

$$\left(q_{n+1}\xi_{n+1} + \frac{1}{\xi_n}\right)\left(q_n\xi_n + \frac{1}{\xi_{n-1}}\right) \leq 1. \tag{7}$$

PROOF. Necessity follows from Theorem 1.1.

To prove sufficiency, define $B_n := q_n\xi_n + 1/\xi_{n-1}$, $S_n := B_{n+1}\xi_n$, and $Q_n := q_n(B_nB_{n+1})^{-1}$. Then $q_n \leq Q_n$ and

$$Q_n S_n + \frac{1}{S_{n-1}} = 1.$$

The result follows from Theorem 4.1 of [7].

COROLLARY 2.2. *Equation (2) is nonoscillatory if there is an integer N such that the inequality*

$$\left(\sqrt{q}_{n+1} + \sqrt{q}_n\right)\left(\sqrt{q}_n + \sqrt{q}_{n-1}\right) \leq 1 \tag{8}$$

holds for $n \geq N$.

PROOF. Let $\xi_n = q_n^{-1/2}$ in Theorem 1.2.

REMARK. Inequality (8) improves the condition $q_n \leq 1/4$ stated in Theorem 1.2.

EXAMPLE. Consider (2) with $\{c_n\}$ and $\{b_n\}$ defined by: $c_{3j+1} = c_{3j+2} = 2$, $c_{3j} = 1$; $b_{2j-1} = 3$, and $b_{2j} = 4\,(j = 1, 2, 3, \dots)$. Then, $\{q_n\}$ is periodically defined by the ordered triple (1/3, 1/3, 1/12), that is, $q_{3j-2} = q_{3j-1} = 1/3$ and $q_{3j} = 1/12$. This sequence satisfies the inequality (8). Therefore, with definition of the sequences $\{b_n\}$ and $\{c_n\}$, (2) is nonoscillatory.

The following notation will be used in the sequel.

NOTATION. For $n \geq m$, we write

$$\bar{q}_n = \sup_{0 \leq k} 4^k q_n q_{n+1} \dots q_{n+k}.$$

Note that

$$\bar{q}_n \geq q_n. \tag{9}$$

LEMMA 2.3. *Suppose that (5) has a positive solution $\{s_n\}$ defined for $n \geq m$. Then, for $m \leq n < \infty$*

$$q_n < \frac{1}{s_n} \leq 1 - \bar{q}_{n+1}. \tag{10}$$

$$q_n + \bar{q}_{n+1} < 1. \tag{11}$$

PROOF. Rewrite (5) as $q_n = 1/s_n(1 - 1/s_{n-1})$. It follows that $q_n < 1/s_n$. On the other hand, $q_{n+1} = 1/s_{n+1}(1 - 1/s_n)$. Therefore,

$$4q_n q_{n+1} = \frac{1}{s_{n+1}} \left[\frac{4}{s_n(1 - 1/s_n)}\right] \left(1 - \frac{1}{s_{n-1}}\right) \leq \frac{1}{s_{n+1}} \left(1 - \frac{1}{s_{n-1}}\right).$$

By induction,

$$4^k q_n q_{n+1} \dots q_{n+k} \leq \frac{1}{s_{n+k}} \left(1 - \frac{1}{s_{n-1}}\right) < 1 - \frac{1}{s_{n-1}}.$$

Therefore, $\bar{q}_n < 1 - 1/s_{n-1}$.

REMARK. In view of (9), inequality (11) improves Theorem 2.3 of [5].

LEMMA 2.4. *If (5) has a positive solution s_n defined for $n \geq m$, then, for any nonnegative sequence α_j and every $k \geq 1$, we have*

$$\sum_{j=m+1}^{m+k} \left(2\sqrt{\alpha_j \alpha_{j+1} q_j} - \alpha_j\right) \leq \frac{\alpha_{m+k+1}}{s_{m+k}} - \frac{\alpha_{m+1}}{s_m}. \tag{12}$$

PROOF. Rewrite (5) as $q_j s_j - 1 = -1/s_{j-1}$. Multiplying by α_j and adding α_{j+1}/s_j, we get

$$\alpha_j q_j s_j + \frac{\alpha_{j+1}}{s_j} - \alpha_j = \frac{\alpha_{j+1}}{s_j} - \frac{\alpha_j}{s_{j-1}}. \tag{13}$$

Since $\alpha_j q_j s_j + \alpha_{j+1}/s_j \geq 2\sqrt{\alpha_j \alpha_{j+1} q_j}$, (13) implies that

$$\left(2\sqrt{\alpha_j \alpha_{j+1} q_j} - \alpha_j\right) \leq \frac{\alpha_{j+1}}{s_j} - \frac{\alpha_j}{s_{j-1}}.$$

Summing the last inequality, we get (12).

Following [8], we define a node of a finite or infinite sequence as follows.

DEFINITION. *For a finite or infinite sequence of real numbers $u : u(a), u(a+1), \dots$, we say that $m = a$ is a node for u if $u(a) = 0$, and we say that $m > a$ is a node for u if $u(m) = 0$ or $u(m-1)u(m) = 0$.*

It should be noted that if a solution x_n of (2) has no nodes between m and $m + k + 1$, then $s_n = (b_{n+1}x_{n+1})/(c_n x_n)$ satisfies (5) for $m \leq n \leq m + k$.

COROLLARY 2.5. *If there is a sequence α_j of nonnegative real numbers such that*

$$\sum_{j=m+1}^{m+k} \left(2\sqrt{\alpha_j \alpha_{j+1} q_j} - \alpha_j\right) \geq \alpha_{m+k+1},$$

then each solution of (2) has at least one node between m and $m+k+1$.

PROOF. In view of the inequality (12), it suffices to see that any solution s_j of equation (5) should satisfy $s_j > 1$.

The following theorem improves Theorem 2.9 of [9], which has a similar statement, but without the rightmost two terms in (14).

THEOREM 2.6. *Equation (1) is oscillatory if there are two sequences ν_k and n_k of positive integers such that $\nu_k > n_k$ and*

$$\sum_{j=n_k+1}^{\nu_k} a_j \geq c_{\nu_k} + c_{n_k} - b_{\nu_k} \bar{q}_{\nu_k} - b_{n_k+1} q_{n_k}. \tag{14}$$

PROOF. Suppose that (2) is nonoscillatory. Then, there is a sufficiently large integer N such that (5) has a positive solution s_j defined for $j \geq N$. Invoking Lemma 2.4, choose $\alpha_j = b_j$. Then, for every m, n $(n > m > N)$, the L.H.S. of (12) (with $n = m+k+1$) can be simplified as follows:

$$\sum_{j=m+1}^{n-1} \left(2\sqrt{\alpha_j \alpha_{j+1} q_j} - \alpha_j\right) = \sum_{j=m+1}^{n-1} (c_j - c_{j-1} + a_j) = c_{n-1} - c_m + \sum_{j=m+1}^{n-1} a_j.$$

On the other hand, (10) implies the following estimate for the R.H.S. of (12):

$$\frac{\alpha_n}{s_{n-1}} - \frac{\alpha_{m+1}}{s_m} < b_n(1 - \bar{q}_n) - b_{m+1} q_m = c_n + c_{n-1} - a_n - b_n \bar{q}_n - b_{m+1} q_m.$$

Therefore, (12) implies that $-c_m + \sum_{j=m+1}^{n-1} a_j < c_n - a_n - b_n \bar{q}_n - b_{m+1} q_m$. That is,

$$\sum_{j=m+1}^{n} a_j < c_n + c_m - b_n \bar{q}_n - b_{m+1} q_m.$$

This contradicts (14).

EXAMPLE. Let $c_n \equiv 1$, $a_{k^2} = a_{k^2+1} = 2/3$, and $a_j = -1$ for $k^2 \neq j \neq k^2+1$. Then, $q_{k^2} = 9/16$, $q_{k^2+1} = q_{k^2-1} = 1/4$, and $q_j = 1/9$ otherwise. One can see that Theorem 2.9 of [9] is not applicable to this example, nor is any other known theorem. However, it is easy to see that with $n_k = k^2 - 1$ and $\nu_k = k^2 + 1$, the L.H.S. of (14) becomes $4/3$ and the R.H.S. becomes $\leq 2 - (4/3) q_{k+1} - (4/3) q_{k^2-1} = 4/3$.

Hence, Theorem 2.6 is applicable and equation (1), with these coefficients, is oscillatory.

COROLLARY 2.7. *If (5) has a positive solution $\{s_n\}$ defined for $n \geq m$, then we have*

$$\sum_{j=m+1}^{m+n} \left(2\sqrt{q_{j+1}} - 1\right) 4q_j < 1 - 4q_m q_{m+1} - \bar{q}_{m+n+2}. \tag{15}$$

PROOF. Choose $\alpha_j = 4q_j$ in Lemma 2.4.

The following theorem bounds the arithmetic mean of the sequence $\sqrt{q}_n$ for nonoscillatory equations.

THEOREM 2.8. *If (2) is nonoscillatory, then there is an integer N such that the inequality*

$$\frac{1}{n}\sum_{j=m+1}^{m+n}\sqrt{q_j} < \frac{1}{2}\left[1+\frac{1-q_m-\bar{q}_{m+n+1}}{n}\right] \tag{16}$$

holds for all $m > N$ and $n \geq 1$.

PROOF. Choose $\alpha_j \equiv 1$ in Lemma 2.4 and use (10) to get

$$\sum_{j=m+1}^{m+n}\left(2\sqrt{q_j}-1\right) < 1-q_m-\bar{q}_{m+n+1}. \tag{17}$$

Now, adding n to both sides of (17) and dividing by $2n$, we get (16).

In fact, the previous theorem bounds the supremum of the arithmetic mean of the sequence $\sqrt{q_n}$ by the upper bound $1/2$, while Theorem 1.3 bounds the supremum of the geometric mean of $\sqrt{q_n}$ by the same upper bound.

To see that Theorem 2.8 improves Theorem 1.3, note that (16) together with (9) imply that

$$\frac{1}{n}\sum_{j=m+1}^{m+n}\sqrt{q_j} < \frac{1}{2}\left[1+\frac{1-(q_m+q_{m+n+1})}{n}\right].$$

Therefore,

$$\left(\sqrt{q_{m+1}}\cdots\sqrt{q_{m+n}}\right)^{1/n} < \frac{1}{2}\left[1+\frac{1-(q_m+q_{m+n+1})}{n}\right].$$

That is,

$$q_{m+1}\cdots q_{m+n} < \frac{1}{4^n}\left[1+\frac{1-(q_m+q_{m+n+1})}{n}\right]^{2n}.$$

Multiplying by $q_m q_{m+n+1}$, one gets

$$q_m\cdots q_{m+n+1} < \frac{1}{4^n}q_m q_{m+n+1}\left[1+\frac{1-(q_m+q_{m+n+1})}{n}\right]^{2n}. \tag{18}$$

Note that the function $f(x,y) = xy[1+(1-(x+y))/n]^{2n}$ defined on the domain $D = \{(x,y) : 0 < x < 1,\ 0 < y < 1\}$ has maximum value $1/4$. Hence, (18) implies that $q_m\cdots q_{m+n+1} < 4^{-(n+1)}$. That is,

$$q_m\cdots q_{m+n} < \frac{1}{4^n}.$$

THEOREM 2.9. *Suppose that q_n is periodic with period 2, that is, $q_{2n} = a$ and $q_{2n-1} = b$. Then (2) is oscillatory if and only if*

$$\sqrt{a}+\sqrt{b} > 1.$$

PROOF. The "if" part follows from Theorem 2.8. The "only if" part follows from Corollary 2.2.

EXAMPLE. Let $q_{2n} = 1/2$ and $q_{2n-1} = 1/8$. This example was treated in [5,7] in two different manners. Alternatively, we may use the foregoing theorem since $\sqrt{1/2}+\sqrt{1/8} = 3/(2\sqrt{2}) > 1$. On the other hand, the inequality (16) is helpful for estimating the number of nodes for such an example since this inequality is valid in any interval $[m, m+n]$ on which a positive solution x_j of (3) exists, while neither of the two expositions in [5,7] would give information of this sort.

EXAMPLE. Let $q_{2n} = 1/2$ and $q_{2n-1} = 1/10$. Theorem 3.2 is applicable to this example since $(\sqrt{1/2}+\sqrt{1/10})^2 - 1 = (\sqrt{5}-2)/5 > 0$. No other criteria seem to cover such an example.

3. COMPARISON THEOREMS

In this section we introduce some new comparison theorems that enable one to compare equations of type (1) with simpler equations of type (3).

THEOREM 3.1. *If equation (1) is nonoscillatory, then so is*

$$\Delta^2 x_{n-1} + \left(\sqrt{q}_n + \sqrt{q}_{n-1} - 1\right) x_n = 0. \tag{19}$$

PROOF. Let $\{s_n\}$ be a positive solution of $q_n s_n + 1/s_{n-1} = 1$. Define $r_n := \sqrt{q}_n s_n$. Then $\{r_n\}$ is a positive solution of

$$\sqrt{q}_n r_n + \frac{\sqrt{q}_{n-1}}{r_{n-1}} = 1.$$

It follows that the equation

$$\Delta\left(\sqrt{q}_n \Delta y_{n-1}\right) + \left(\sqrt{q}_n + \sqrt{q}_{n-1} - 1\right) y_n = 0$$

is nonoscillatory. Since $\sqrt{q}_n < 1$, the result follows from the celebrated Sturm comparison theorem.

THEOREM 3.2. *Suppose that*

$$q_n s_n + \frac{1}{s_{n-1}} = 1,$$

$$q'_n s'_n + \frac{1}{s'_{n-1}} = 1$$

have positive solutions $\{s_n\}$ and $\{s'_n\}$, defined for $n \geq N$. Then,

$$\sqrt{q_n q'_n} S_n + \frac{1}{S_{n-1}} = 1$$

has a positive solution $\{S_n\}$.

PROOF. Since

$$\left(q_n s_n + \frac{1}{s_{n-1}}\right)\left(q'_n s'_n + \frac{1}{s'_{n-1}}\right) = 1$$

can be expanded as

$$q_n q'_n s_n s'_n + \frac{1}{s_{n-1} s'_{n-1}} + \frac{q_n s_n}{s'_{n-1}} + \frac{q'_n s'_n}{s_{n-1}} = 1$$

and

$$2\sqrt{\frac{q_n q'_n s_n s'_n}{s_{n-1} s'_{n-1}}} \leq \frac{q_n s_n}{s'_{n-1}} + \frac{q'_n s'_n}{s_{n-1}},$$

hence,

$$\left(\sqrt{q_n q'_n}\sqrt{s_n s'_n} + \frac{1}{\sqrt{s_{n-1} s'_{n-1}}}\right)^2 \leq 1.$$

Now, let $\xi_n \equiv \sqrt{s_n s'_n}$. Then,

$$\left(\sqrt{q_n q'_n}\xi_n + \frac{1}{\xi_{n-1}}\right) \leq 1.$$

The required result now follows from Theorem 2.1.

THEOREM 3.3. *If (1) is nonoscillatory, then so is*

$$\Delta^2 x_{n-1} - (q_n^{-1/2} - 2)x_n = 0. \tag{20}$$

PROOF. Applying Theorem 3.2 with $q_n' = q_{n+1}$, we find that the equation $\sqrt{q_n q_{n+1}} S_n + 1/S_{n-1} = 1$ has an eventually positive solution S_n. Letting $R_n = \sqrt{q_{n+1}} S_n$, we find that

$$R_n + \frac{1}{R_{n-1}} = q_n^{-1/2}.$$

According to Theorem 1.1, this implies that (20) is nonoscillatory.

COROLLARY 3.4. *Equation (1) is oscillatory if any of the two conditions*

1. $\limsup_{n\to\infty} \sum_{j=m}^{n} (q_j^{-1/2} - 2) = \infty$,
2. $\liminf_{n\to\infty} \sum_{j=m}^{n} (q_j^{-1/2} - 2) < \limsup_{n\to\infty} \sum_{j=m}^{n} (q_j^{-1/2} - 2)$.

PROOF. With the aid of Theorem 3.3, the first part follows from Corollary 3.3 of [7] and the second follows from Corollary 2.8 of [9].

REMARK. The foregoing theorem improves Theorem 2.1 of [5] whenever the sequence q_n satisfies the condition $q_n + q_{n+1} \le 3/4$, which is the case for most of the interesting examples. To see this, compare $(q_n^{-1} - 1)(q_{n+1}^{-1} - 1)$ with $q_n^{-1/2} q_{n+1}^{-1/2}$. Note that $q_n + q_{n+1} \le 3/4$ implies that $1 - (q_n + q_{n+1}) \ge 1/4$. On the other hand, $1/4 \ge \sqrt{q_n q_{n+1}}(1 - \sqrt{q_n q_{n+1}})$. Therefore, $(1 - q_n)(1 - q_{n+1}) > \sqrt{q_n q_{n+1}}$.

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